Physics: Introduction to Electromagnetic theoryname Subject code: BSC-PHY-101G

ECE/ME

Ist Semester

Unit 4: Electromagnetic waves

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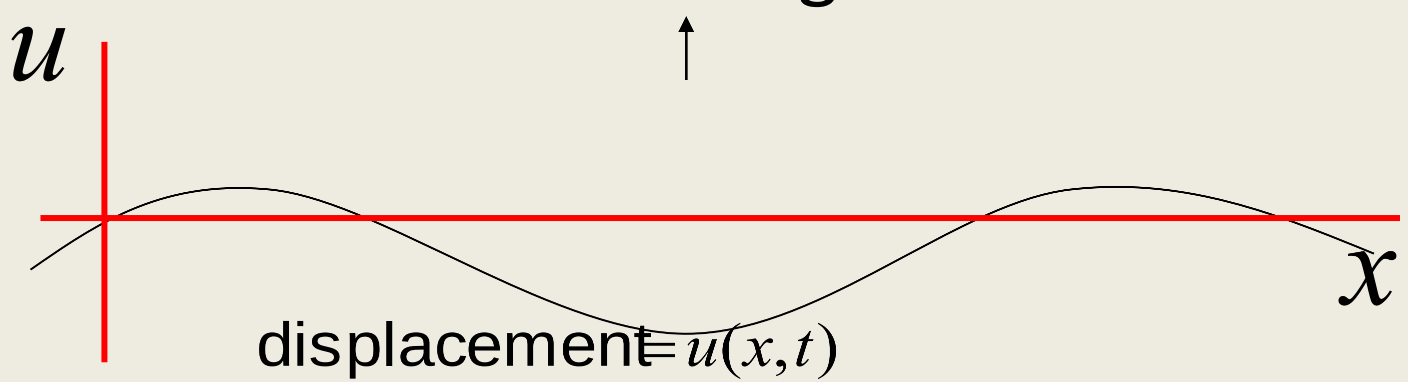
The Wave Equation

Motion of a string

Imagine that a stretched string is vibrating.

The wave equation says that, at any position on the string, acceleration in the direction perpendicular to the string is proportional to the curvature of the string.



The one-dimensional wave equation

Let

- x = position on the string
- t = time
- $u(x, t)$ = displacement of the string at position x and time t.
- T = tension (parameter)
- ρ = mass per unit length (parameter)

Then

$$\rho \frac{\partial^2}{\partial t^2} u(x,t) = T \frac{\partial^2}{\partial x^2} u(x,t)$$

Equivalently, $\boxed{u_{tt} = a^2 u_{xx}}$ where $a = \sqrt{T/\rho}$

Solving the one-dimensional WE

First, we make a wild assumption: suppose that u is a product of a function of x and a function of t:

$$u(x,t) = X(x)T(t)$$

Then the wave equation becomes

$$X(x)T''(t) = a^2 X''(x)T(t)$$

So,

$$\frac{T''(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)}$$

The only way for this equation to be true **for all** x and for all t is for both sides to be constant; that is, $T''(t)/a^2T(t) = \lambda$ and $X''(x)/X(x) = \lambda$.

Separation of variables

Now $$\frac{T''(t)}{a^2T(t)} = \frac{X''(x)}{X(x)} = \lambda$$

means that $$T''(t) - \lambda a^2 T(t) = 0,$$ and
$$X''(x) - \lambda X(x) = 0,$$

If $\lambda < 0$, $T(t)$ and $X(x)$ are both trig functions.*

*If $\lambda = 0$, they're linear, and if $\lambda > 0$, they're exponential. But let's not worry about that now. **(Explanation on next slide.)**

Clarification: the sign of λ

There are three types of solutions to the equation

$$X''(x) - \lambda X(x) = 0$$

1. if $\lambda = 0$, then $X(x)$ is linear ($ax + b$), which won't satisfy the boundary conditions;
2. if $\lambda > 0$, $X(x)$ is exponential ($ke^{\sqrt{\lambda}t}$), which also won't satisfy the boundary conditions; and
3. if $\lambda < 0$, then $X(x)$ is a sinusoid $X(x) = k\sin\left(\left(\sqrt{-\lambda}\right)x\right)$ that satisfies the boundary conditions. This is why

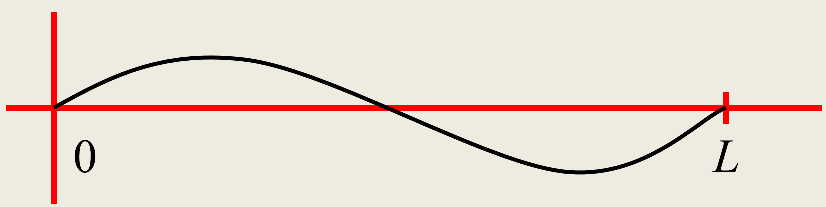
$\lambda = -\dfrac{n^2\pi^2}{L^2}$ in slide 8.

Boundary conditions

In a stringed instrument, each end of the string is fixed; if the string has length *L*, then, for all *t*,

$$u(0,t) = X(0)T(t) = 0, \text{ and}$$

$$u(L,t) = X(L)T(t) = 0.$$



Since $T(t) \neq 0$ for all *t*, $X(0) = X(L) = 0$; thus, $X(x)$ could be a sum of sine terms with zeros at $x = L$:

$$X(x) = k\sin(n\pi x/L)$$

Finding *T(t)*

Now we know two things:

$$X''(x) - \lambda X(x) = 0 \qquad\qquad X(x) = k\sin(n\pi x / L)$$

This means that

The **minus** sign is correct. The original slide was wrong.

$$\lambda = -\frac{n^2\pi^2}{L^2}$$

Now substitute λ into the equation for *T*:

$$T''(t) - \lambda a^2 T(t) = 0$$

And find the solution:

$$T(t) = b\sin\frac{an\pi t}{L} + c\cos\frac{an\pi t}{L}$$

Putting it all together

We now have a family of solutions for the wave equation (one for every *n*):

$$u(x,t) = X(x)T(t) = \sin\frac{n\pi x}{L}\left(b\sin\frac{an\pi t}{L} + c\cos\frac{an\pi t}{L}\right)$$

Suppose we choose an x_0 and look at how the displacement varies at this point as *t* changes.

$$u(x_0,t) = K\left(b\sin\frac{an\pi t}{L} + c\cos\frac{an\pi t}{L}\right)$$

$$\left(K = \sin\frac{n\pi x_0}{L}\right)$$

This is the equation of a sinusoid with frequency = $\dfrac{an}{2L}$

where $n = 1, 2, 3\ldots$

Take it higher

So there are three ways to increase the frequency of a sound, producing a "higher" note:

Increase *a*
(continuous changes)
Tuning

Increase *n*
(discrete changes)
Overblowing
Playing harmonics

$$\text{frequency} = \frac{an}{2L}$$

Decrease *L*
(continuous changes)
Shortening
Fretting

Boundary conditions

With wind instruments, it's not always true that if L is the length of the tube, $u(0, t) = u(L, t) = 0$.

This *is* true with the flute (because the pressure doesn't change at the ends).

However, what happens with some instruments (like the sax) is that $u_x(0, t) = u_x(L, t) = 0$, which means that the $X(x)$ functions are cosine terms rather than sines.

Still more bizarre is the behavior of the clarinet. It has boundary conditions

$$u(0,t) = 0$$

$$u_x(L,t) = 0$$

Go back and substitute in these boundary conditions to see what happens!

Standing waves

The harmonics we hear on a stringed instrument and the overtones on a wind instrument are actually produced by **standing waves**, which are solutions of the form

$$b \sin \frac{n\pi x}{L} \cos \frac{an\pi t}{L}$$

Sketch some standing waves (see demo on the web).

How are the standing waves for a clarinet different from those for a flute?

Exercises

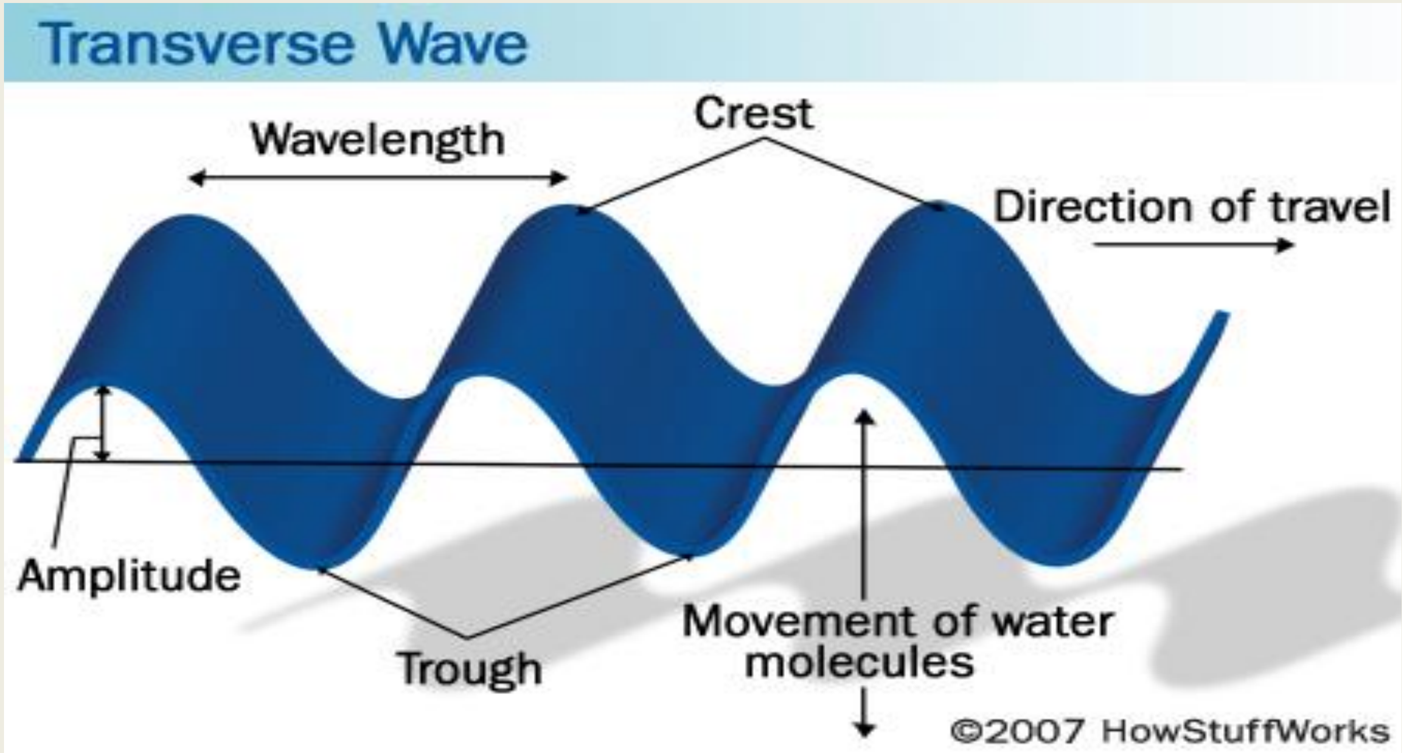
1. Verify that the sum of any two solutions to the wave equation is also a solution.
2. Redo the wave equation solution using the boundary conditions for a flute: $u_x(0, t) = u_x(L, t) = 0$
3. Redo the wave equation solution using the boundary conditions for a clarinet: $u(0, t) = u_x(L, t) = 0$. Find the frequencies of the solutions, and sketch the standing waves that are solutions to this equation.
4. Use separation of variables to find a family of solutions to the heat equation:

$$u_t(x,t) = ku_{xx}(x,t)$$

with boundary conditions $u(0, t) = u(L, t) = 0$.

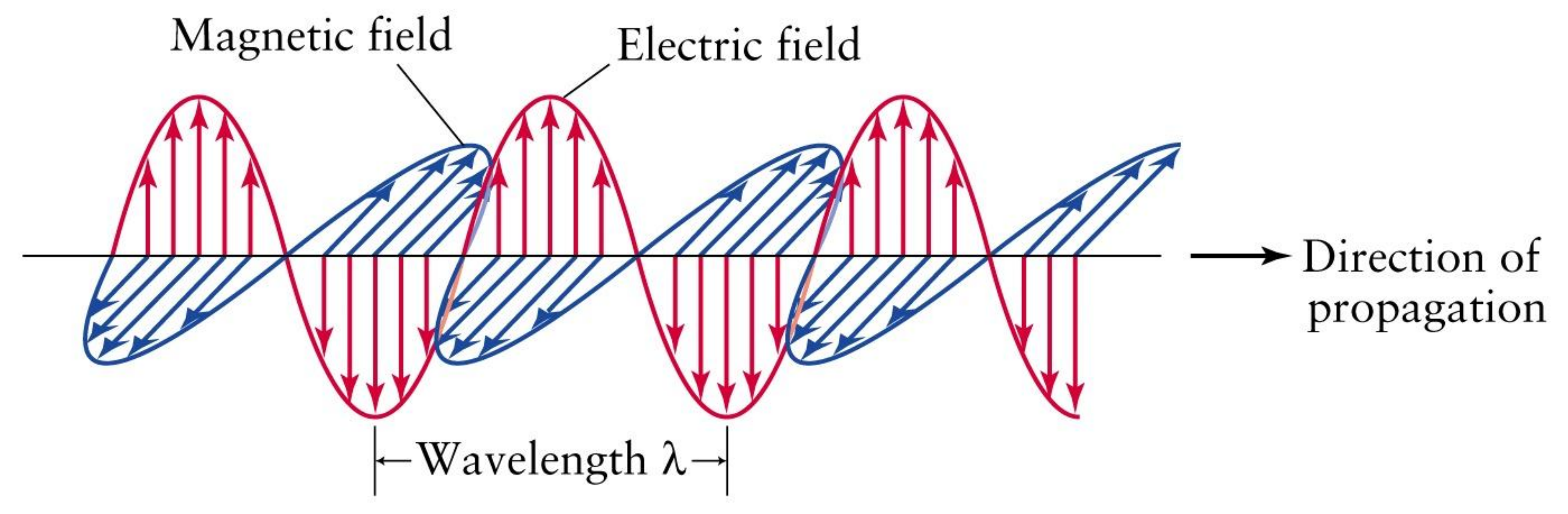
Electromagnetic Waves

- **Transverse waves without a medium!**
- (They can travel through **empty space**)

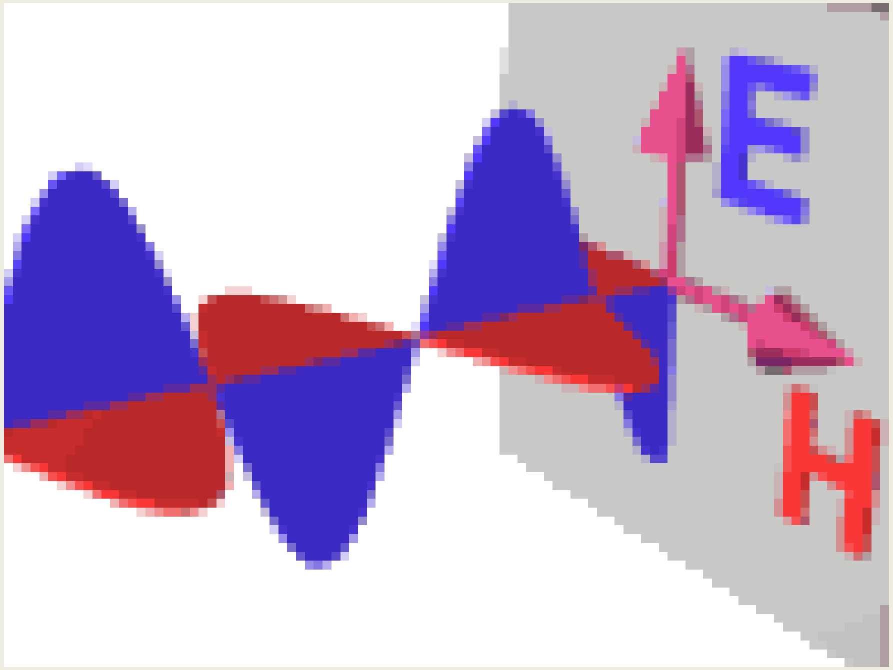


- They travel as **vibrations in electrical and magnetic fields.**

- Have some magnetic and some electrical properties to them.



- When an electric field changes, so does the magnetic field. The changing magnetic field causes the electric field to change. When one field vibrates—so does the other.
- **RESULT**-An **electromagnetic wave.**



Deriving EM waves in vacuum

In vacuum, $\rho = 0$, $\vec{J} = \vec{0}$:

$$\vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \vec{0} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Apply *curl* ($\vec{\nabla}\times$) to the third equation:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B})$$

$$\Rightarrow -\nabla^2 \vec{E} + \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B})$$

Apply the first and the fourth equations:

$$\Rightarrow -\nabla^2 \vec{E} + \vec{\nabla}(0) = -\frac{\partial}{\partial t}\left(\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$$

$$\Rightarrow \nabla^2 \vec{E} = \underbrace{\mu_0 \varepsilon_0}_{\frac{1}{c^2}} \frac{\partial^2 \vec{E}}{\partial^2 t}$$

$$\Rightarrow \nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial^2 t}, \quad \text{where } c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

This is the wave equation in 3D, c.f. $\dfrac{\partial^2 y}{\partial^2 x} = \dfrac{1}{v^2} \dfrac{\partial^2 y}{\partial^2 t}$ in 1D.

Future Scope and relevance to industry

- https://www.sciencedirect.com/journal/wave-motion/articles-in-press
- https://www.sciencedirect.com/science/article/pii/S0022247X02001725
- https://www.researchgate.net/publication/299476836_On_the_Transverse_Vibrations_of_Strings_and_Beams_on_Semi-Infinite_Domains

NPTEL/other online link

- https://nptel.ac.in/courses/122107037/12
- \https://nptel.ac.in/courses/112104227/18
- https://library.seg.org/doi/abs/10.1190/1.3603649
- http://farside.ph.utexas.edu/teaching/em/lectures/node90.html
- https://nptel.ac.in/courses/115101005/downloads/lectures-doc/Lecture-33.pdf